Tesis para optar al grado de Master en Economía

Título:

“On the Relationship between Awareness and Completeness in Possibility Models”

Autor: Esteban Javier Peralta

Mentor: Dr. Fernando Tohmé

Victoria – Buenos Aires

Febrero 2012
On the Relationship between Awareness and Completeness in Possibility Models

Esteban Peralta
Department of Economics, Yale University
peraltaej@hotmail.com

Abstract

Brandenburger ([5]) introduced the notion of possibility models and showed that, under very weak conditions, a complete model does not exist; i.e., a well-defined possibility set not represented within the model always exists. This paper builds on that result, proving that unawareness of a non-empty event is a necessary and sufficient condition for incompleteness in a possibility model.

JEL Code: C72, D80, D83
Keywords: incomplete possibility models, knowledge, language, unawareness.

Clearly, the incompleteness of type spaces has a direct connection with unawareness.

Zhen Liu, 2007

1 Introduction

One of the fundamental questions in the analysis of a decision-making situation is how much do an agent know about the model itself. This question is not only interesting from the philosophical point of view, but also crucial for the correct definition of the model. Analysts are mostly concerned with the characterization of the “world” about which the agent builds her beliefs. But then, a complete and exhaustive description of that world should include those beliefs as well. After all, a state of the world is such only if the agent thinks that it is a possible state of the world [26]. So, a circular argument arises, since every world must contain a description of the possibilities conceived by the agent.
Of course, Game Theory, being a representation of mutual externality situations, is not an exception. On the contrary, every interactive situation makes things much worse, since the beliefs of any agent about anything relevant for her must include a belief about the beliefs of the others, a belief about the beliefs of the others about her own beliefs, and so on. Hence, a recursive process of belief formation arises. This process, emerging from the agents themselves, may lead to a class of well-defined potential worlds. The agents, in turn, can use this information to make decisions and, in this way, actualize one of those worlds.

Even if the process is easily describable in words, it is formally complex, since it involves an infinite sequence of beliefs. The literature has been looking for different ways of representing such sequences in order to simplify the analysis. This search began with the work of Harsanyi ([13]), which introduced the concept of type structure as a model of the hierarchies of beliefs. By means of types the analysts could get rid of explicit sequences of beliefs.

However, such a description must be “appropriate”, not only in terms of tractability, but also in the sense that it must provide the same information as the process itself. As Friedenberg ([11]) points out:

Yet, for a type structure to be a model, it should be able to represent all hierarchies of beliefs. That is, the analyst should not lose any hierarchies by using the model.

This is the idea of completeness introduced in [5]. That is, a model is complete if every possible belief is represented, i.e. every believable element is believed. After Harsanyi the literature became more and more interested in studying the conditions that have to be fulfilled by a model in order to be complete. This is not surprising since completeness is an essential requirement for most of the solution concepts we usually use. Unfortunately, despite many conditions have been found, under which the use of a model does not cause a loss of generality, it has been shown that such conditions do not always hold [5],[8],[26]. This means that not every possible belief can be actually represented in a model dealing with infinite sequences of beliefs. Thus, there might exist some potential beliefs that are not held by the agents. Since beliefs refer to worlds, there might be some worlds that are not conceived by the agents.

The literature on Epistemic Game Theory has offered a specific knowledge-related meaning for conceivability, namely, awareness. It is said that an agent is aware of some event if she knows the event or she knows that she does not

---

1 See [6] for an overview.
2 [21], [7], [14] and [11], among others
know it. Thus, *not to conceive an event* is interpreted as the complement of *being aware* of it. That is, not to know and not to know that it is not known. As a consequence, unawareness is related with a complete lack of knowledge.

Unawareness appears to be really important in several economic situations, especially through the players' recognition of the possibility of being unaware of something [16]. This seems natural, since agents usually face unforeseen contingencies (think, for instance, in contract theory). What seems to be underlying such phenomena is that, if an agent might be unaware of something, the ways in which she can can think are limited. If an agent is unaware of some worlds (conceived by the analyst), she is not only incapable of conceiving such worlds by herself, but also incapable of conceiving whether the other agents conceive them. But then, the question arises about the beliefs that an agent may form about the behavior of the other agents, about the beliefs of the others about her own beliefs, and so on. Since these worlds do not cover the full range of possibilities, there will be worlds missing in the model.

Accordingly, it seems very likely that, mediated by the circular nature of states of the world, *completeness* and *awareness* could be related in some way. Since both completeness and unawareness can be associated to a certain *lack of conception* a natural conjecture is that the same lack of conception underlies both phenomena. More precisely, we can argue that incompleteness means that there are worlds of which the players are unaware and, conversely, that if a player is unaware of some event there might exist beliefs which are not represented within the model.

While completeness and unawareness have been extensively studied in the literature, previous research seems to have missed the question of their possible connection. Thus, *if, how and to what extent* they are related are the main concerns of the present paper. Our main claim can informally be described as:

*A possibility model is incomplete if and only if there exists an agent who is unaware of some non-empty set of states of the world.*

The rest of the paper is organized as follows. The next section deals with a brief description of the framework we will use throughout the paper: *possibility models*. They can be seen as non-probabilistic analogues of type models. Section 3 introduces the notion of completeness proposed in [5] to show that, under suitable conditions, a complete possibility model does not exist.

By introducing a formal definition of unawareness, section 4 presents the main results of the paper. Namely, that under certain non-trivial conditions, a
possibility model is incomplete if and only if we find a set of states of the world about which the agents are unaware of. The paper concludes with sections 5 and 6, where some conceptual matters and related work, are respectively discussed.

2 Possibility Models: the Brandenburger-Keisler Framework

Suppose the existence of a group of agents facing uncertainty about the elements of a set $S$ of states of nature. Usually, $S$ represents the payoff-relevant variables in a game (payoff-functions, strategies, etc.). By definition, in every interactive situation each agent must form a belief, not only about the elements of $S$, but also about each other's beliefs about $S$, about each other's beliefs about each other's beliefs about $S$, and so on. This gives rise to the so called explicit approach, where each belief level is expressed explicitly, resulting in a hierarchy of beliefs. This leads to a rather cumbersome structure of infinite sequences of beliefs. Many researchers have chosen, instead, to adopt an implicit approach to the representation of beliefs.

Possibility models, first introduced in [5], are interactive models of the implicit kind. They are basically the non-probabilistic analogues of type models. Like these, they are useful to deal, implicitly, with the mutual beliefs of a group of agents facing uncertainty about their environment. For a formal presentation, we follow the treatment given in [8]. For simplicity, we assume a two-player situation ($I = \{a, b\}$).

Definition 1 A Possibility Model is a structure

$$\mathcal{P} = (R^a, R^b, P^a, P^b)$$

where $R^a$ and $R^b$ are non-empty sets and $P^a$ and $P^b$ are proper subsets of $R^a \times R^b$ and $R^b \times R^a$, respectively.

Members of $R^a$ (resp. $R^b$), denoted by $x$ (resp. $y$), are called states of $a$ (resp. $b$). Members of $R^a \times R^b$, denoted by $w = (x, y)$, are called states of the world. That is, the states of the world are identified with a profile of states of the agents.

---


5We take $x, x', x'', \ldots$ as representing elements of $R^a$ while $y, y', y'', \ldots$ do the same for elements in $R^b$. 
Let $\Omega = R^a \times R^b$ be the set of states of the world and will call every $E \subseteq \Omega$ an event. Usually, events are the primitive object of uncertainty. $P^a \subseteq \Omega$ and $P^b \subseteq \Omega^-$ are called possibility relations and relate states of one agent with states of the other. Thus, for any $x$, $P^a(x) = \{(x, y) : (x, y) \in P^a\} \subset \Omega$ denotes the possibility set of $x$ (likewise for $b$). That is, it yields the states of the world that are conceived possible in the state $x$ of agent $a$. This means, in turn, that at state $x$ agent $a$ conceives possible a family of states of $b$, namely those $y$'s such that $(x, y) \in P^a(x)$.

The possibility relations are assumed to be serial, i.e. $\forall x \exists y \text{ s.t. } P^a(x) \neq \emptyset$ and $\forall y \exists x \text{ s.t. } P^b(y) \neq \emptyset$. Thus, at every state of an agent some states of the other are considered possible.

The serial nature of the possibility relations allows to define, for any $x \in R^a$, the $x$-section of $P^a$ on $R^b$, $P^a_x = \{y \in R^b : (x, y) \in P^a\}$. By a slight abuse of language we say that a state of an agent “assumes” a set of states of the other player:

**Definition 2** A state $x \in R^a$ assumes a non-empty set $Z \subseteq R^b$, if $P^a_x = Z$.

That is, $a$ assumes $Z$ in state $x$ if the set of states of $b$ that $a$ considers possible at $x$ coincides with $Z$. Therefore, since the possibility relations are serial, every state for $a$ (resp. $b$) assumes a subset of $R^b$ (resp. $R^a$).

Following the convention in [7] and [3] the players are supposed to “know their own state”. That is, for any event $E \subseteq \Omega$ and any state $x$ of $a$, the set of states in $E$ that $x$ should not rule out is $\{(x', y') \in E : x' = x\}$. So, let

$E_x = \{y' \in R^b : (x, y') \in E\}$

This allows us to define a knowledge operator $K^a : 2^\Omega \rightarrow 2^\Omega$ for $a$:

**Definition 3** Agent $a$ knows $E$ in state $(x, y)$ if $P^a_x = E_x$. Therefore, we let $K^a(E) = \{(x, y) : P^a_x = E_x\}$ be the set of states of the world in which $a$ knows $E$ (likewise for $b$).

That is, agent $a$ knows an event $E$ in every one of the states in which she assumes the projection of $E$ over $R^b$. We incorporate this notion (absent in [8]) because it provides a way of extending the concept of assumption to states.

---

6. To simplify the notation we will denote $\Omega^- = R^b \times R^a$.

7. Note that we can informally interpret a state of a player as a player’s type. In fact, this is the case of Strategic Belief Models ([8], section 8).

8. This is our formalization of the idea that the players should be self-conscious [18]. For a discussion, see sections 2 and 7.c of [1], and section 7 in [28].
of the world, allowing the application of possibility models to the analysis of unawareness, as it will be shown in the next sections.

Notice that the definition of the knowledge operator defined here has other properties than those usually associated to the concept [23]:

- \(K^a(E) \subseteq E\) for any non-empty event \(E \subseteq \Omega\). This means that \(K^a\) violates the so-called Axiom of Knowledge. The reason is that by definition \(K^a(E) = \{x \in R^a : P^a_x = E_x\} \times R^b\), so, for any \((x, y) \in K^a(E)\), unless \(E_x = R^b\), \((x, y) \notin E\).
- \(K^a(\Omega) \subseteq \Omega\) (likewise for \(b\)). Otherwise, it would mean that for every \(x \in R^a\), \(P^a_x = R^b\), i.e. \(P^a = \Omega\), contrary to the assumption that \(P^a \subset \Omega\).
- On the other hand we have that \(K^a(\emptyset) = \emptyset\). This is because \(K^a(\emptyset) = \{x \in R^a : P^a_x = \emptyset\} \times R^b\) where \(\{x \in R^a : P^a_x = \emptyset\} = \emptyset\).
- Given two events \(E, F\), \(E \subseteq F\) does not imply \(K^a(E) \subseteq K^a(F)\). That is, the knowledge operator violates monotonicity. To see this, consider a case in which \(E \subseteq F\) and so there exists an \(x\) such that \(E_x \subset F_x\). Then, either \(P^a_x = E_x\) or \(P^a_x = F_x\) but not both. In particular, if \(P^a_x = E_x\), then \((x, y) \in K^a(E)\) but \((x, y) \notin K^a(F)\).
- \(K^a\) (as well as \(K^b\)) does not satisfy the Axiom of Transparency. That is, for any event \(E\), \(K^a(E) \not\subseteq K^a(K^a(E))\). Consider all the states \((x, y)\) in which \(a\) knows \(E\). They are such that \(K^a(E) = \{x \in R^a : P^a_x = E_x\} \times R^b\). The \(x\)-section of this class yields \(R^b\) which in general is different from \(E_x\). Therefore, if \((x, y) \in K^a(E)\), in general, \((x, y) \notin K^a(K^a(E))\).
- Instead, \(K^a\) violates the Axiom of Wisdom. That is, \(\Omega \setminus K^a(E) \not\subseteq K^a(\Omega \setminus K^a(E))\). It is easy to see that since \(K^a(E) = \{x \in R^a : P^a_x = E_x\} \times R^b\), \(\Omega \setminus K^a(E) = \{x \in R^a : P^a_x \neq E_x\} \times R^b\). If for any \((x, y) \in \Omega \setminus K^a(E)\) we take its \(x\)-section, we obtain \(R^b\), which in general is different from \(P^a_x\). Therefore, if \((x, y) \in \Omega \setminus K^a(E)\), in general, \((x, y) \notin K^a(\Omega \setminus K^a(E))\).

### 3 Complete Possibility Models

5.6 The limits of my language mean the limits of my world

*Ludwig Wittgenstein, 1922*

---

9As we will see in section 4, this assumption, already advanced by [8], is critical for our formal argument.
In the previous section we indicated that possibility models are useful non-probabilistic settings that deal, implicitly, with infinite sequences of beliefs. Therefore, they must represent the same information as the explicit approach. Otherwise, the implicit approach may lead to a loss of information since some beliefs could get left out of the model. Given that every state of the world is a description of the beliefs of all the agents in the game, the aforementioned loss of information may induce a parallel loss of some of the hierarchical sequences that may obtain by unfolding the beliefs held by the agents. The key question, then, is whether it is possible to construct a model in which each state of the world is an exhaustive description of the state of nature and of the players’ beliefs [17]. In other words, while we know that, by construction, every state of the world induces a hierarchy of beliefs starting from about \( S \), we are wondering about the converse. That is, does each hierarchy of beliefs arise from (i.e., can be represented in) a state of the world of the model? If we find a model that contains every hierarchy we will say that it is complete.

Unfortunately, [5], [8] and [26] showed that, under very weak conditions, a complete possibility model does not exist. That is, it is always possible to find a hierarchy of beliefs that cannot be represented within the model. Thus, there must exist potential beliefs that no agent can hold.

Among these conditions, perhaps the most important is given by the features of what [5] and [8] call the language of the players. Briefly, the language can be identified with the class of expressions about objects and the uncertainty surrounding them. It restricts the way in which the agents can think about everything relevant for them. As pointed out in [8] (pg. 8):

*Given a belief model, the next step is to specify a language used by the players to think about beliefs. We’ll then be able to talk about the completeness of a model, which is relative to a language.*

That is, we need to specify how the players think before we can say whether the language is appropriate for the expression of their thoughts. This is the sense in which completeness is relative to a language. For instance, the language chosen in probabilistic settings (e.g. in type structures) is usually given by the set of Borel probability measures. Thus, the objects of uncertainty are the Borel-measurable events only, instead of all possible events [18]. This indicates that the language consists of the \( \sigma \)-algebra of sets imposed on the model.

A language \( \mathbf{L} = \mathbf{L}^a \times \mathbf{L}^b \), is a pair of languages, one for each player. But before we define the language to be used, we first need to give the formal definition of a complete model.
Definition 4 Let $P$ be a possibility model and let $L$ be a language for $P$. $P$ is complete for $L$ if for each $Z \in L$ there exists $x \in R^a$ such that $P^a_x = Z$ and for each $W \in L$ there exists $y \in R^b$ such that $P^b_y = W$.

In words: a possibility model is complete if every non-empty possibility set expressible in the language of a player is assumed by a state of the other player.

Here, in particular, we choose the language provided by the power set operator:

\[ L^\mathcal{P} = \{ L = L^a \times L^b : L^a \subseteq \wp(R^a) \text{ and } L^b \subseteq \wp(R^b) \}. \]

where, given a set $X$, its power set is $\wp(X)$ with cardinality $2^{\lvert X \rvert}$. We can distinguish here two subclasses:

\[ L^\mathcal{P}_c = \{ L = L^a \times L^b : L^a \subset \wp(R^a) \text{ and } L^b \subset \wp(R^b) \}. \]

and

\[ L^\mathcal{P}_e = \{ L = L^a \times L^b : L^a = \wp(R^a) \text{ and } L^b = \wp(R^b) \}. \]

In all this cases we focus on the family of subsets in $R^b$ that $a$ can conceive as well on those in $R^a$ conceivable by $b$. A possibility model built on $L^\mathcal{P}$ (as well as on $L^\mathcal{P}_c$ or $L^\mathcal{P}_e$) is complete if every non-empty set of states of $b$ that belongs to $\wp(R^b) \setminus \emptyset$ is assumed by one state of $a$ (the same is true for every non-empty set of states of $a$).

To see the limits of completeness in possibility models consider the following family of languages:

\[ L^\mathcal{P}^+ = \{ L = L^a \times L^b : L^a \subseteq \wp(R^a), L^b \subseteq \wp(R^b), \text{ where } \lvert L^a \rvert > \lvert R^b \rvert + 1 \text{ or } \lvert L^b \rvert > \lvert R^a \rvert + 1 \}. \]

We have that:

**Proposition 1** No possibility model $P$ is complete for a language $L \in L^\mathcal{P}^+$.

**Proof:** Consider a language $L$ for which, without loss of generality, $\lvert L^a \rvert > \lvert R^b \rvert + 1$. A function $p : R^b \to L^a$ can be defined, such that $p(y) = W \in L^a$, where $W = P^b_y$. But, because of the difference between the cardinalities of $R^b$ and $L^a$ this means that there has to exist a non-empty $W \in L^a$ for which no $y \in R^b$ satisfies $p(y) = W$.

The key condition for the proof is the difference in cardinalities. By a straightforward application of Cantor’s Theorem, we have that:\[10\]

---

\[ ^{10} \text{For a discussion of positive results, see section 10 of [8].} \]
Corollary 1 ([8], Proposition 5.1) No possibility model $P$ is complete for a language $L \in \mathcal{L}_\mathcal{P}$.

This result cannot be extended to $\mathcal{L}_\mathcal{P}^\omega$.

Example: Consider $R^a = R^b = \{x, y, z\}$ with

$$P^a = \{(x, \{x, y\}), (y, \{x, y\}), (z, \{x, y\})\}$$

and

$$P^b = \{(x, \{x, z\}), (y, \{x, z\}), (z, \{x, z\})\}$$

If the individual languages are $L^a = \{W\}$ and $L^b = \{Z\}$, where $W = \{x, z\}$ and $Z = \{x, y\}$ we have that $L = L^a \times L^b \in \mathcal{L}_\mathcal{P}^\omega$ but the model $P$ is complete. To see this, consider the non-empty set expressible in $L^a$, $W$. At any state in $R^b$, say $y$ we have that $P^b_y = W$. The same is true for $Z$ in $L^b$.

We are here interested on shedding some light on the possible causes and consequences of the existence of incomplete structures. Although [8] and [26] emphasize on the possibility of holding negative self-referential beliefs as the cause of incompleteness we will focus, instead, on the agents being unaware of some non-empty events.

4 Unawareness and Completeness

The standard approach to reasoning about knowledge assumes, implicitly, that agents are aware of all the relevant aspects of the problem at hand. But in real life situations agents usually have to deal with unforeseen contingencies. To model such situations, the characterization of the agents has to make explicit the assumption that they are not necessarily aware of all the relevant features of the problem [12].

The literature usually relates unawareness to knowledge [22]. Formally, we will say that $a$ is unaware of an event $E$ if she does not know $E$ and does not know that she does not know $E$. Thus, let $U^a : 2^\Omega \rightarrow 2^\Omega$ be the unawareness operator defined by $U^a(E) = \Omega \setminus K^a(E) \cap \Omega \setminus (K^a(\Omega \setminus K^a(E)))$ such that $U^a(E)$ represents the set of states of the world at which $a$ is unaware of $E$. Let $\Omega \setminus U^a(E) = A^a(E)$ be the set of states in which $a$ is aware of $E$ (likewise for $b$).

We focus on possibility models satisfying the following axioms:\footnote{For a different approach see [12].} \footnote{For justifications see [4], [9], [12] and [15] among others.}
(KU) \textit{KU Introspection}: for every $E \subseteq \Omega$, $K^aU^a(E) = \emptyset$.

(AU) \textit{AU Introspection}: for every $E \subseteq \Omega$, $U^a(E) \subseteq U^aU^a(E)$.

(UK) \textit{UK Introspection}: for every $E \subseteq \Omega$, $U^a(E) \subseteq U^aK^b(E)$.

(S) \textit{Symmetry}: for every $E \subseteq \Omega$, $U^a(E) = U^a(\Omega \setminus E)$.

(SR) \textit{Self-Reflection}: for every $E \subseteq \Omega$, $A^a(E) = A^aK^a(E)$.

The same must also be true for $b$.

These axioms are now standard in the literature and their interpretation is very straightforward. So, if a model $\mathcal{P}$ satisfies KU introspection then, if an agent is unaware of an event, it cannot exist a state of the world in which the agent knows that she is unaware. AU introspection captures a similar intuition: if an agent, say $a$, is unaware of an event $E$, $a$ has to be unaware of the event consisting of the states of world in which $a$ is unaware of $E$. In other words, unawareness is a notion that refers to a complete lack of every level of positive knowledge [9].

On the other hand, both symmetry and self-reflection are intuitive additional requirements. Symmetry indicates that an agent cannot be unaware of an event without being unaware of its complement while self-reflection indicates that an agent who is unaware of some event must be unaware that she knows the event.

Finally, the model includes a description of what agents may know about each other’s knowledge. This is accomplished by UK introspection, which establishes that if an agent is unaware of an event, she cannot be aware that the other agent knows it.

The next results show that incompleteness and unawareness are indeed related in a non-trivial way. To this end, we first show that if a possibility model is incomplete (for any power set language), there must be events of which the agents are unaware. Then, we show that, under suitable conditions, the converse is also true.

\textbf{Proposition 2} \textit{Suppose that} $\mathcal{P}$ \textit{is an incomplete possibility model with language} $\mathcal{L} \in \mathcal{L}^{o+}$. \textit{Then, there exist non-empty events} $Z$ \textit{and} $H$ \textit{such that} $U^a(Z) \neq \emptyset$ \textit{and} $U^b(H) \neq \emptyset$.

\textbf{Proof}: \textit{Since} $\mathcal{P}$ \textit{is incomplete, there exists} $A \in \mathcal{L}^a$ \textit{that no} $y$ \textit{assumes or} $B \in \mathcal{L}^b$ \textit{that no} $x$ \textit{assumes. Suppose, without loss of generality, that the latter is the case}. Pick an arbitrary $x$ such that $\{x\} \times B = Z$. \textit{By construction,} $Z_x = B$ \textit{and} $Z_{x'} = \emptyset$ \textit{for every} $x' \neq x$. \textit{It follows that} $K^a(Z) = \{(x'',y''): P^a_{x''} = Z_{x''}\}$. 

10
Since for any \( x'' \neq x \) we have that \( Z_{x''} = \emptyset \), the only remaining possibility is that \( x'' = x \). But then, this means that, since \( P^a \) is serial, \( P^a_x = B \), but this means that \( x \) assumes \( B \). Contradiction. That is, \( K^a(Z) = \emptyset \) and consequently we have (i): \( \Omega \setminus K^a(Z) = \Omega \).

On the other hand, since \( P^a \subset \Omega \) we have that \( K^a(\Omega) \subset \Omega \). Hence, there must exist at least one world \( \omega \in \Omega \setminus K^a(\Omega) \). Substituting with (i), we obtain (ii): \( \omega \in \Omega \setminus K^a(\Omega \setminus K^a(Z)) \). On the other hand, by (i) again we have trivially that (iii): \( \omega \in \Omega \setminus K^a(Z) \). Then, from (ii) and (iii) it follows that \( \omega \in \Omega \setminus K^a(\Omega \setminus K^a(Z)) = U^a(Z) \).

The case when no \( y \) assumes the non-empty set \( A \) is treated analogously.

In words: if a possibility model is incomplete, there must exist a non-empty possibility set of some player (for some language) that is not assumed by any state. Thus, there must be an event that, by construction, is not known by that player.\(^{13}\) On the other hand, since the players do not know the space of states of the world at every state of the world, there must be a state in which the same player does not know that she does not know the event. That is, she is unaware of the event.

Notice that this result is independent of the specific elements of the languages of the players.\(^{14}\) It indicates that whenever a possibility model is incomplete, some agent will necessarily be unaware of some event. Therefore, it suggests an interpretation of the relation between incompleteness and unawareness. It indicates that unawareness, like incompleteness, obtains due to the properties of the power sets of \( R^a \) and \( R^b \).

Up to this point we have shown that incompleteness leads to unawareness. Let us see that the converse is also true:

**Proposition 3** Consider a possibility model \( \mathcal{P} \) that satisfies KU introspection and suppose that there exist two non-empty events \( Z \) and \( H \) such that \( U^a(Z) \neq \emptyset \) and \( U^b(H) \neq \emptyset \). Then, there must exist a language \( L \in \mathcal{L}^{\omega^+} \) such that \( \mathcal{P} \) is incomplete with respect to \( L \).

**Proof:** Without loss of generality suppose that \( U^a(Z) \neq \emptyset \) for some non-empty \( Z \subseteq \Omega \). Let \( U^a(Z) = B \). By KU introspection, \( K^a(B) = \emptyset \). Then, for every \( x \in R^a \), \( P^a_x = B_x \). Notice that, since \( B \neq \emptyset \), there must exist some \( \bar{x} \in R^a \) such

\(^{13}\) The choice of \( \{x\} \) in the proof is completely arbitrary. Any subset of \( R^a \) yields the same result. The point is that it is enough to show that some agent is unaware of some event. See section 5.b.

\(^{14}\) Provided that the languages are defined as subsets of the power set.
that $B_x \neq \emptyset$, Thus, since $B_x \subseteq R^b$ there exists a language $L \in \mathcal{L}^{\mathbb{P}}$ such that $B_x \in L^b$ even if $\bar{x}$ does not assume $B_\bar{x}$. Then, $\mathcal{P}$ is incomplete for $L$.

Notice that the intuition is, again, really simple. If an agent, say $a$, is unaware of an event $E$, KU introspection ensures that $a$ will not know in any state of the world the event $A = [a \text{ is unaware of } E]$. Therefore, we have a possibility set $A_x \neq \emptyset$ that belongs to some language but is not assumed by any of $a$’s states. Thus, the model is incomplete.

This conclusion is not surprising. After all, if there are languages that make a player unaware of some events, given any event about which a player is unaware, one of those languages, in which that event can be defined but not known, can be found.

Another but equivalent way to refer to completeness relies on the possible existence of holes in the model [8]. That is, sets of states of one player that cannot be assumed by any state of the other. Formally:

**Definition 5** A possibility model $\mathcal{P}$ with a language $L \in \mathcal{L}^{\mathbb{P}}$ has a hole at a set $D \subseteq R^a$, expressible in the language $L^a$, if $D \neq \emptyset$ and no $y \in R^b$ assumes $D$ (analogously exchanging the roles of $a$ and $b$).

A model $\mathcal{P}$ is complete with respect to a language $L \in \mathcal{L}^{\mathbb{P}}$ if and only if $\mathcal{P}$ has no holes in $L^a$ and $L^b$.

Considering the axioms on unawareness, Proposition 3 seen in the light of the existence of holes provides another perspective for the origin of incompleteness in possibility models. The axioms intend to convey the idea that if a player is unaware of some event $E$, she must be unaware of many others. For example, of the event of being unaware of $E$. Accordingly, the next proposition uses these axioms to show that if an agent is unaware of some (non-empty) events, the model has to have holes.

**Proposition 4** Consider a possibility model $\mathcal{P}$ satisfying KU introspection, AU introspection, UK introspection, symmetry and self-reflection with a language $L$, such that there are non-empty sets $Z \subseteq \Omega$ and $H \subseteq \Omega$ such that $U^a(Z) \neq \emptyset$ and $U^b(H) \neq \emptyset$. Then, there must exist a language $L \in \mathcal{L}^{\mathbb{P}}$ for which $\mathcal{P}$ is incomplete.

**Proof:** Suppose that $U^a(Z) \neq \emptyset$ for some non-empty $Z \subseteq \Omega$. We will show that, by a straightforward application of the axioms on $K^a$ and $U^a$, a language can be defined in which holes may be found.

\[\text{(1)}\] Of course, by Proposition 1, no model with a language in $\mathcal{L}^{\mathbb{P}}$ is complete.
Let $U^a(Z) = A$. Thus $A_x \neq \emptyset$ for some $x \in R^a$. By KU introspection $K^a(A) = \emptyset$. Then, for any $x \in R^a$, $P^a_x \neq A_x$. Let $F = A_x$. By definition, $F \subseteq R^b$. Therefore, if $F \in L^b$, no $x \in R^a$ assumes $F$. That is, $P$ has a hole at $F$.

On the other hand, from AU introspection, we get that $U^aU^a(Z) \neq \emptyset$. Call this event $B$. Thus $B_x \neq \emptyset$ for some $x \in R^a$. By KU introspection, $K^a(B) = \emptyset$. Then, for any $x \in R^a$, $P^a_x \neq B_x$. Let $G = B_x$. Thus, if $G \in L^b$, no $x \in R^a$ assumes it. That is, $P$ has a hole at $G$.

By UK introspection, $U^aK^b(Z) \neq \emptyset$. Let $C$ be this event. Thus, $C_x \neq \emptyset$. Again KU introspection ensures that $K^a(C) = \emptyset$. Then, for every $x \in R^a$, $P^a_x \neq C_x$. Let $H = C_x$. Thus, if $H \in L^b$, no $x \in R^a$ assumes it.

By self-reflection $U^aK^a(Z) \neq \emptyset$. Let $D$ be this event. Thus, $D_x \neq \emptyset$ for some $x \in R^a$. By KU introspection $K^a(D) = \emptyset$. But then, for every $x$, $P^a_x \neq D_x$. Let $J = D_x$. Then, if $J \in L^b$, no $x \in R^a$ assumes it.

Finally, by symmetry we have $U^a(\Omega \setminus Z) \neq \emptyset$. Let $E$ be this event. Thus, $E_x \neq \emptyset$. By KU introspection, $K^a(E) = \emptyset$. For every $x \in R^a$, $P^a_x \neq E_x$. Let $M = E_x$. Again, if $M \in L^b$, no $x \in R^a$ assumes it.

This means that if $L^b$ includes any of the sets $F$, $G$, $H$, $J$ or $M$, the model is incomplete w.r.t. $L$ (the same argument ensues for $b$).

## 5 Concluding remarks

This paper focused on the possible relation between the concepts of incompleteness and unawareness. While each one has received, separately, extensive attention, their possible connection has not been thoroughly studied. This seems a bit odd since both problems share a common origin. Namely, the existence of limits in the belief formation processes of agents. We intend this paper to provide a baseline result that might start a line of research on the relation between both approaches.

We have shown that, subject to some usual conditions, a relation between the incompleteness of a model and the existence of non-trivial unawareness indeed exists. Put it differently, under certain conditions, unawareness is both necessary and sufficient for incompleteness. This was achieved by showing that there exists an intrinsic relation between the language of the players and the lack of conception that underlies the notion of unawareness. That is, our results suggest that the awareness or not of some events depends crucially on the language they use. Thus, we show that, like incompleteness, unawareness is a language-dependent property. This is not surprising. On the contrary, it really fits with
the intuition that a player’s language restricts the class of events that she may know. In other words, players can think only through their languages [8].

6 Discussion

There are some topics that deserve further consideration:

a. Assumptions and Beliefs: Here we used the term assumption as in [8]. But, unlike there, we do not define an explicit notion of belief. Since we deal with possibility correspondences, one may interpret the possibility set considered possible at a state as the “belief” of such state. But we find more useful to define beliefs explicitly in order to show that “assumptions” (and not “beliefs”) are the key element in our result. As in [8], we might say that the state $x$ of $a$ believes a set $Y \subseteq R^b$ if (likewise for $b$) $P^a_x \subseteq Y$. Thus, $x$ believes the sets which contain all the states that $x$ considers possible. Therefore, assumes (strictly) implies believes. Accordingly, it is worth to note that while $a$ does not assume $\Omega$ at every state of the world, she indeed believes $\Omega$ in every state of the world. Therefore, it is perfectly possible that an agent, being unaware of an event, still believes it.

b. Unawareness: It is worth noting that in this paper we focused on the particular definition of unawareness given in [22], in which the lack of conception of the players is predicated exclusively on knowledge. The literature, however, has identified other notions that, in different settings, can be more appropriate (e.g. [12]). The differences in the characterization of unawareness do not affect the fact that unawareness may arise non-trivially as a feature of well-defined possibility structures. In this sense, our result is in line with some previous results in the literature. For instance, in [9] it is shown that partitional information structures are incompatible with unawareness. This is due to the important role of the necessitation axiom in that setting (see [25] for an additional discussion on this point). In our paper we have not imposed a structure on the knowledge operator, except for KU and UK Introspection. Furthermore, necessitation is false in our model. Nevertheless, unawareness arises in these two disparate settings just because of the structure of the possibility model.

c. States of the players and states of the world as objects of knowledge: This paper assumes that the primary objects of uncertainty are
states of the players. That is, the states assume states of the other players, but players themselves know events in $\Omega$. Since we take for granted the self-conscious assumption, a state’s knowledge encompasses also classes of states of the world. We could have considered alternative definitions in which the states of the players consider possible only states of the world, just by adding relations to the language (like in [8]). For instance, we could define relations over $U^a \times U^a \times U^b$ and $U^b \times U^a \times U^b$ such that each state of each agent is related to a set of states of the world. Thus, both states and players have the same domain of knowledge.

d. Languages and terminal objects: The analysis in this paper has emphasized on the class of languages $\mathcal{L}^{\nu+}$ which ensure the incompleteness of the possibility model. The question is whether the results reported here ensue in the case of other languages. In very general terms it can be said that the language defines a category of objects and a functor among them. In our case, sets and their powerset, but it can be topologies and functions, spaces of measurable sets and functions among them, etc. The idea is to consider each object as a “type” and the functor as yielding the beliefs that may be held by the type. Each choice of type model (in our case a possibility model) is called a coalgebra. Completeness is ensured by the existence of a terminal object (or final coalgebra) in the category [24] [20].

In this sense the restriction to power set languages does not seem excessively restrictive. The language of open or of compact sets can also be defined in terms of the power set. Provided that a cardinality condition is satisfied, analogous to the one in the definition of $\mathcal{L}^{\nu+}$, a similar result follows. Therefore, all the results presented should apply also in those settings, provided the same conditions are satisfied, namely that $P^a \subset \Omega$, $P^b \subset \Omega^-$ and the cardinality of events exceeds the number of states of the world. But this not the case when the category is that of measurable spaces. There, as shown in [19] there exists a final coalgebra, while for the categories based on $\mathcal{L}^{\nu+}$ no final coalgebra exists. The situation changes if one replaces the power set functor with the functor that assigns to each set the set of all its finite non-empty subsets [27]. This makes sense: the reason why Proposition 1 is true is that the cardinality of the possibility sets is strictly larger than that of the sets on which they are conceived. Restricting attention to finite non-empty sets destroys that result.
References


